CONIC SECTIONS

29

ELEMENTS

OF THE

CONIC SECTIONS.

BY

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PREFACE.

For two years I have been giving my advanced classes in Geometry work in conic sections by means of notes copied by the hektograph. The amount of work has grown on my hands till it has become too large for convenient use in its present form; so it is now put in the hands of the printer.

To show the connection of the curves by their definition, and not to follow too closely the line of reasoning in other American works on this subject, I used Boscovitch's definitions, and find by so doing many theorems can be demonstrated with less work than by using the ordinary definitions.

The demonstrations are necessarily longer than those relating to the straight line and circle, but there is a corresponding advantage in discipline to the student.

Although more details are given than in English works, it is hoped that enough has been omitted to require the independent thought of the student; and the author trusts it may be of some use to those desiring a knowledge of this extensive subject.

Kingston, Pa.,
Jan. 28, 1887.

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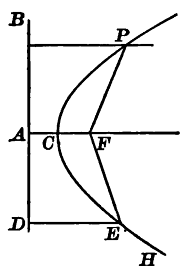
DEFINITIONS.

A curved line is a line no part of which is straight, and points on the curve are so situated that they answer the requirements of some definition. To define the curved lines called conic sections, we assume a fixed straight line, called a *directrix*, and a fixed point without this line, called a *focus*.

A conic section is a curved line the distance from any point in which to the directrix is in a constant ratio to its distance from the focus.

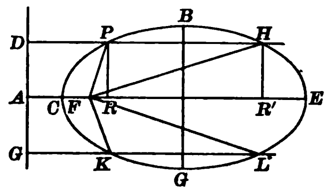
There are three classes of curves coming under this definition.

The parabola, in which any point is the same distance from the directrix as from the focus; *i.e.*, the constant ratio of the above definition equals unity.



Take BD as the directrix, and F as the focus of the parabola PCH; then $BP = PF$ and $DE = EF$.

The ellipse, in which any point is farther from the directrix than from the focus; *i.e.*, the constant ratio is greater than unity.

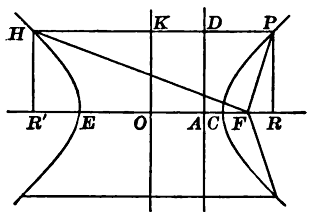


Take DG as the directrix, and F as the focus; then

$$\frac{DP}{PF}=\frac{DH}{HF}=\frac{GK}{KF}=\frac{GL}{LF},$$

and $$DP > PF.$$

The hyperbola, in which any point is nearer the directrix than the focus; *i.e.*, the constant ratio is less than unity.



Take DA as the directrix, and F the focus of the hyperbola $PHEC$; then

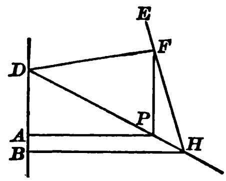
$$\frac{DP}{PF}=\frac{DH}{HF},$$

and $$DP < PF.$$

PROBLEM I.

Having given a point on a line, to find a second point on the line such that the ratio of its distances from a fixed line

(the directrix) and from a fixed point (the focus) shall equal the ratio of the corresponding distances from the given point.



Let DB be the directrix, and F the focus. Take P as the given point on the line DH, and suppose the problem solved, and H to be the required point; then

$$AP : PF :: BH : HF; \tag{1}$$

or, by alternation,

$$AP : BH :: PF : HF; \tag{2}$$

AP and BH being perpendicular to DB by construction,

$$AP : BH :: DP : DH; \tag{3}$$

comparing (2) and (3),

$$PF : HF :: DP : DH;$$

therefore DF bisects the angle PFE.

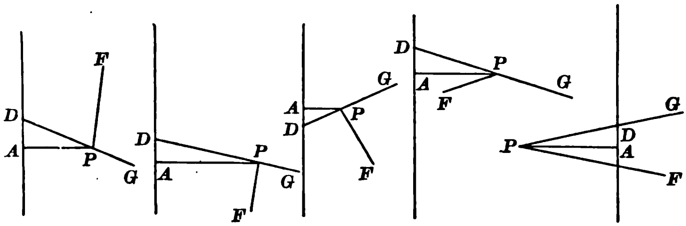
Hence, to find the required point, join the focus F with the point D in which the line intersects the directrix, and draw a line making an angle DFE equal to DFP: the point in which this line EF meets the given line is the required point.

Corollary I. As but one line can be drawn from F, making with DF an angle equal to DFP, there are but two points on a line from which a given ratio, formed as in the problem, can be made.

Corollary II. If the point P be moved along the line DH so that DFP approaches a right angle, its equal DFE

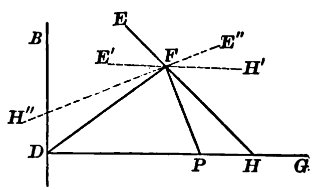
also approaches a right angle, and when each becomes a right angle, the line EFH coincides with FP, and H with P; hence there will be but one point on a line from which a given ratio can be formed when the line joining this point to the focus is perpendicular to the line joining the focus to the intersection of the given line and the directrix.

Exercises. Find a second point on the line DG of each of the following figures, so that the ratio of the lines corresponding to AP and PF shall equal their ratio.



Theorem I.

A line perpendicular to the directrix will meet a parabola in one point, it may meet an ellipse in two points, and it will meet a hyperbola in two points.



Take BD as the directrix, F the focus, and P a point of a conic section on the line DG perpendicular to BD. Make $DFE = DFP$, to find the second point of the conic on DG (Prob. I.).

(1) If P is a point of a parabola $DP = PF$ and

$$PDF = PFD = DFE;$$

hence EF will be parallel to DG, and there is no second point. As F is without the directrix, a point P may be found on any perpendicular to BD such that $PD = PF$ by erecting a perpendicular to the middle of DF; hence every perpendicular to the directrix intersects a parabola.

(2) If P is a point of an ellipse $DP > PF$; hence

$$PDF < PFD = DFE,$$

and EF is not parallel to DG, but will intersect it in the direction DG. If the line DG be moved farther from F, the angle PDF will increase; and as $PFD > PDF$, their sum will finally exceed two right angles, when PF would meet DG on the opposite side from DF, in which case PD could not be greater than PF; hence a perpendicular to the directrix may pass above or below the ellipse.

(3) If P is a point of a hyperbola $DP < PF$; hence

$$PDF > PFD = DFE,$$

and FE must meet DG on the opposite side of the directrix from the focus.

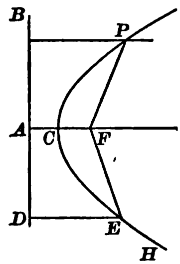
Since $PFD < PDF$, the triangle DPF is always possible, and there will always be two points in which a line perpendicular to the directrix intersects a hyperbola; but these points are on opposite sides of the directrix.

Theorem II.

All the conic sections are symmetrical with regard to a line drawn from the focus perpendicular to the directrix.

From the focus F draw FA perpendicular to the directrix DB; then FA is an axis of symmetry. For draw BP per-

pendicular to BD, and intersecting the curve in P. Take $AD = AB$ and draw DE perpendicular to BD. Revolve



the upper portion of the figure about AF as an axis, and BP will coincide with DE, P coming at E, such a point that

$$DE : EF = BP : PF,$$

or at a point of the curve. If BP intersects the curve in two points, the second point would come on such a point of DE as answers the definition of a conic section.

Definitions. The perpendicular from the focus upon the directrix is called the *principal axis* of the conic section, and the point in which it intersects the curve is the *vertex*.

A distance measured from the vertex, or any specified point, along the principal axis is called an *abscissa; e.g.*, CG, Fig. Prob. II. A line perpendicular to the principal axis and meeting the conic is an *ordinate; e.g.*, PG.

A tangent to a conic section is a straight line that meets the curve in one point without intersecting it. If a line which intersects a conic in two points be revolved about one of these points till the other coincides with it, the line will then be a tangent.

A subtangent is that portion of the principal axis included between the point in which a tangent meets the axis and the foot of an ordinate drawn from the point of tangency; *e.g.*, EG, Fig. Prob. II. A normal is a line drawn perpendicular to the tangent from the point of tangency; *e.g.*, PK.

A subnormal is that portion of the principal axis included between the point in which the normal intersects the axis and the foot of the ordinate from the point of tangency; *e.g.*, GK.

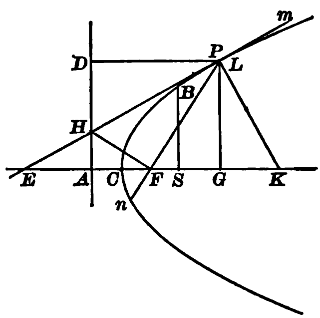
A parameter is the double ordinate through the focus. The parameter of a parabola equals twice the distance from the focus to the directrix.

A line drawn from the focus to a point of a conic section is called a *focal radius*.

Problem II.

To draw a tangent to a parabola from a given point on the parabola.

Let P be the given point. Join P and the focus F. From F draw FH perpendicular to FP and join P and H. The line PH will have its two points of intersection with the parabola coincident (Prob. I., Cor. II.); hence it is a tangent.



Corollary I. The right triangles DPH and HPF have $DP = PF$ and HP common; hence they are equal, and the angle DPH equals HPF; or the tangent at any point of a parabola bisects the angle between a line perpendicular to the directrix and a line to the focus, both being drawn from the point of tangency.

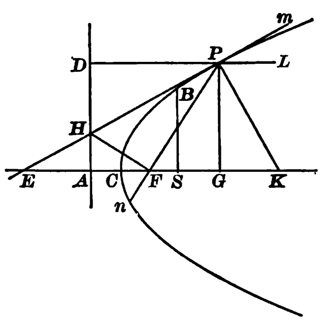
COROLLARY II. Since $PEF = EPD = EPF$, $PF = FE$; or, the distance from the intersection of a tangent to a parabola with the principal axis, to the focus, equals the distance from the focus to the point of tangency.

COROLLARY III. Since $AG = DP = FP = FE$, if we take $AC = CF$ from the first and last of these equals, $CG = EC$; or, the subtangent is bisected at the vertex.

COROLLARY IV. Since nH is a tangent, tangents at the extremities of a chord through the focus meet on the directrix.

THEOREM III.

The subnormal of a parabola is constant and equal to the distance from the focus to the directrix.



Let PK be the normal at the point P; then $GK = AF$.

From the right angles EPK and mPK take EPF and its equal mPL respectively,

and $\qquad FPK = KPL = PKF$;

hence $\qquad KF = PF = AG$.

Taking out the common part FG, $GK = AF$.

Corollary I. In the right triangle EPK, $\overline{PG}^2 = EG \cdot GK = 2\, CG \cdot AF$, or the ordinate at any point of a parabola is a mean proportional between the abscissa of the point and the parameter.

Corollary II. Similarly,

$$\overline{BS}^2 = 2\, CS \cdot AF,$$

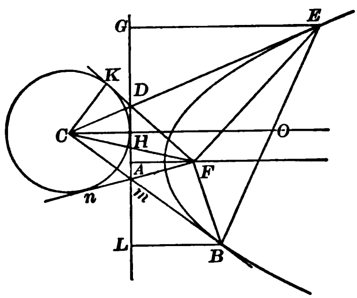
and

$$\overline{PG}^2 : \overline{BS}^2 :: GC : SC;$$

or the squares of any two ordinates are to each other as their abscissas.

Problem III.

From a point without a parabola, to draw a tangent to the parabola.



Let C be the given point. Draw CH perpendicular to AH, and with C as a centre, and CH as a radius, describe a circle. From F draw FK a tangent to this circle, and produce it, if necessary, to D on the directrix. Connect D and C; then DCE is a tangent. For, draw EF perpendicular to DF, and EG perpendicular to AG, from the similar triangles we get

$$DC : DE :: CK : EF :: HC : GE.$$

But $CK = CH$; hence $EF = GE$, and E must be on the parabola; and DE is a tangent, since DFE is a right angle.

COROLLARY I. Draw Fn tangent to the circle KHn, and the tangent mB to the parabola.

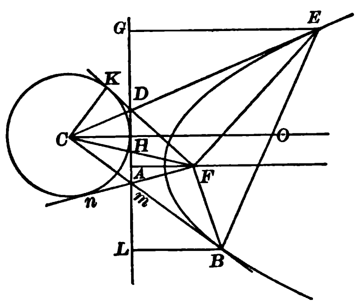
Since $CFn = CFK$, if we add these angles to the right angles BFm and DFE, $CFB = CFE$; or, a line drawn from the intersection of two tangents to a parabola to the focus makes equal angles with the focal radii drawn to the points of tangency.

COROLLARY II. Since DE is the perpendicular bisector of FG, $CF = CG$; likewise $CF = CL$. This will give a method of drawing a tangent to a parabola. Draw a circle from the point C as a centre, with a radius CF; at the points G and L, where it intersects the directrix, erect perpendiculars to meet the parabola in E and B: these points will be the points of tangency.

DEFINITION. A line from a point on a parabola parallel to the principal axis is called a diameter of the parabola.

THEOREM IV.

A diameter drawn through the intersection of two tangents to a parabola bisects the chord joining the points of contact.

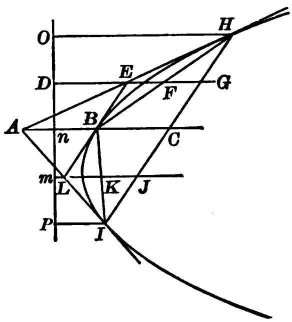


The line CO parallel to AF drawn from C, the intersection of the tangents CB and CE, bisects the chord EB.

Let D be the point in which CE intersects the directrix, the right triangles DFE and DGE being equal, $DF = GD$. As FK is a tangent to the circle CH (Prob. III.), $DK = DH$; hence $GH = FK$. In like manner, $HL = Fn$. Since $Fn = FK$, $HL = HG$, and HO, parallel to GE and LB, bisecting GL, also bisects BE.

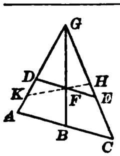
Theorem V.

If from a point on the diameter of a parabola extended two tangents be drawn, the chord of contact is parallel to the tangent at the extremity of the diameter.



From A, on the diameter BC, draw the tangents AH and AI, also the tangent LBE and the chord HI; then HI is parallel to LE.

Draw the chords BH and BI, also EG and LJ parallel to AC. C is the middle of HI, F of BH, and K of BI (Th. IV.); hence, $On = nP$, $OD = Dn$, $mn = mP$, and Dn, the half of On, equals mn, the half of nP; then, in the trapezoid $DELm$, $BE = BL$. Since AC in the triangle AHI bisects the base HI, and also the line LE, LE must be parallel to HI.*



* Let GB be the medial line of the triangle AGC, and let $DF = FE$. If DE is not parallel to AC, take KH through F parallel to AC; then $KF = FH$, and the triangles DFK and HFE will be equal. These being equal, angle DKF equals angle FHE, and DA and HC are parallel, which is impossible since they are the sides of a triangle.

Corollary I. All chords joining the points of contact of pairs of tangents intersecting on the same diameter are parallel, since they are all parallel to the tangent at the extremity of the diameter.

Corollary II. A diameter of a parabola bisects all chords parallel to the tangent through its extremity.

Corollary III. Since

$$BE = CG = GH,$$

and $$LB = JC = JI, \ LE = \tfrac{1}{2} HI.$$

Corollary IV. Since

$$BF = HF, \ AE = EH, \text{ and } AB = BC;$$

or, the vertex of any diameter bisects that part of the diameter included between a chord parallel to the tangent at the extremity of the diameter and the intersection with a tangent through the extremity of the chord.

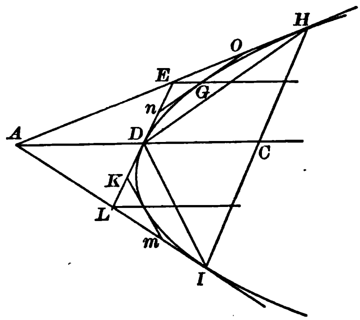
Theorem VI.

The area included between a parabola and a chord is two-thirds of the triangle formed by the chord and the tangents at its extremities.

The area $DIH = \frac{2}{3} AIH$. The triangle $ALE = \frac{1}{4} AHI$, as LE bisects the lines AH and AI; likewise, any tangent at the extremity of a diameter will cut off one-fourth of the triangle formed by a parallel chord and tangents at its extremities.

E being the middle of AH, $ADE = \frac{1}{2} ADH = EDH$, and $EnO = \frac{1}{4} EDH = \frac{1}{4} ADE$; likewise, $LKm = \frac{1}{4} ADL$, and the two exterior triangles EnO and LKm are together

$$\tfrac{1}{4} ALE = \tfrac{1}{16} AHI.$$



Continuing this process indefinitely we should find the area included between AH, AI, and the curve equal

$$\tfrac{1}{4}AHI + \tfrac{1}{16}AHI + \tfrac{1}{64}AHI + \text{etc.} = \tfrac{1}{3}AHI;$$

hence the area between the chord and the curve is

$$AHI - AHDI = \tfrac{2}{3}AHI.$$

Corollary. If the tangents intersect on the principal axis produced, the area between the curve and chord of contact is two-thirds the rectangle of the abscissa and double ordinate of the point of tangency.

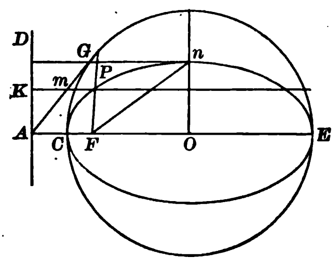
THE ELLIPSE.

Take F as the focus, and AD the directrix of an ellipse; there will be two points in which the principal axis intersects the ellipse C and E (Th. I.).

DEFINITION. The line CE is called the major axis of the ellipse; O, the centre of the axis, the centre of the ellipse. The perpendicular On is the minor semi-axis of the ellipse.

THEOREM VII.

All points of an ellipse are within a circle drawn on its major axis as a diameter.



Describe a circle on the diameter CE. Take P any point on the ellipse; then P is within the circle. For, by definition,

$$AC : CF = AE : EF,$$

and AF is divided harmonically, and F is within the circle. We know, by Geometry, that the circle on the diameter CE is the locus of all points whose distances from A and F are in the ratio $AC : CF$; hence

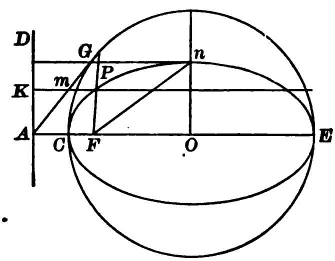
$$AC : CF = AG : GF = Am : PF = KP : PF;$$

hence $Am = KP$. But $Am > Km$, Km being a perpendicular; therefore $KP > Km$; or P on the line GF is farther from the directrix than m on the line AG, or P is between F and G; that is, within the circle.

Definition. The circle drawn with the major axis of an ellipse as a diameter is called the circumscribed circle of the ellipse.

Theorem VIII.

Half the major axis is a mean proportional between the distances from the centre to the directrix and to the focus.



CO is a mean proportional between AO and FO. For, by definition,

$$AC : CF = AE : EF;$$

by the principles of proportion,

$$AC : CF :: AC + AE : CF + EF$$
$$:: AE - AC : EF - CF.$$

But $\quad AC + AE = 2\,AC + CE = 2\,AC + 2\,CO = 2\,AO;$

likewise, $\quad CF + EF = 2\,CO, \quad AE - AC = 2\,CO,$

and $\quad EF - CF = 2\,FO;$

hence $\quad AC : CF :: AO : CO :: CO : FO.$

COROLLARY I. Let n be a point of the ellipse on the perpendicular through the centre. It is seen by the above demonstration that

$$Dn : nF = AO : CO.$$

But $Dn = AO$; hence $Fn = CO$; or, the distance from the focus to the extremity of the minor axis of an ellipse equals half the major axis.

COROLLARY II. Join D and F. Since

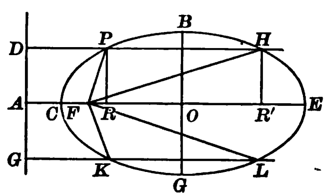
$$Dn : Fn = CO : FO = Fn : FO,$$

and $$DnF = nFO,$$

the triangles nDF and nFO are similar, and DFn is a right angle and Dn is a tangent; hence the minor axis is the longest ordinate of an ellipse.

THEOREM IX.

Ordinates of an ellipse equally distant from the centre are equal.



Take $$OR = OR';$$

then $$PR = HR'.$$

Let $$DP : PF = n;$$

then $$PF = \frac{DP}{n} = \frac{AR}{n} = \frac{AO - RO}{n} = \frac{AO}{n} - \frac{RO}{n}.$$

By Theorem VIII.

$$DP : PF = AO : CO = CO : FO = n;$$

hence $CO = \dfrac{AO}{n}$, and $FO = \dfrac{CO}{n}$.

$$\therefore\ PF = CO - \frac{RO}{n},\ RF = FO - RO = \frac{CO}{n} - RO.$$

From the right triangle RPF

$$PR^2 = PF^2 - RF^2 = \left(CO - \frac{RO}{n}\right)^2 - \left(\frac{CO}{n} - RO\right)^2$$

$$= \frac{n^2-1}{n^2}(CO^2 - RO^2).$$

Since H is on the ellipse,

$$DH : HF = n;$$

or, $$HF = \frac{DH}{n} = \frac{AR'}{n} = \frac{AO}{n} + \frac{OR'}{n} = CO + \frac{OR'}{n};$$

$$FR' = FO + OR' = \frac{CO}{n} + OR'.$$

From the right triangle FHR' we get, as before,

$$\overline{HR'}^2 = \overline{HF}^2 - \overline{FR'}^2 = \frac{n^2-1}{n^2}(\overline{CO}^2 - \overline{OR'}^2).$$

Comparing this value of HR' with that already found for PR, we see they are equal.

Theorem X.

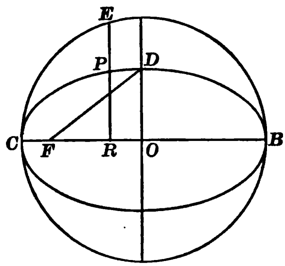
An ellipse is symmetrical with respect to its minor axis.

Revolving an ellipse about its minor axis, the student will see that from Theorem IX. it answers the definition of a symmetrical figure.

THEOREM XI.

An ordinate of the ellipse is to the corresponding ordinate of the circumscribed circle as the minor axis is to the major axis.

$$PR : ER :: OD : OC.$$



From Theorem IX.

$$PR^2 = \frac{n^2-1}{n^2}(\overline{CO}^2 - \overline{RO}^2).$$

From Geometry

$$ER^2 = CR \cdot RB = (CO - RO)(CO + RO) = \overline{CO}^2 - RO^2,$$

and

$$PR^2 : ER^2 = \frac{n^2-1}{n^2}(\overline{CO}^2 - RO^2) : \overline{CO}^2 - \overline{RO}^2 = n^2 - 1 : n^2.$$

Since $\quad FD = CO$

and $\quad \dfrac{FD}{FO} = \dfrac{n}{1},$

or, $\quad \overline{FD}^2 : \overline{FO}^2 = n^2 : 1,$

we get by division,

$$\overline{FD}^2 - \overline{FO}^2 : \overline{FD}^2 :: n^2 - 1 : n^2.$$

But $\quad \overline{FD}^2 - \overline{FO}^2 = \overline{DO}^2.$

Substituting equals for equals,

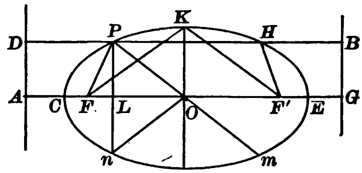
$$\overline{PR}^2 : \overline{ER}^2 :: DO^2 : \overline{CO}^2;$$

or, $\quad PR : ER :: DO : CO.$

COROLLARY. The squares of ordinates to an ellipse are to each other as the rectangles of the segments into which they divide the major axis.

PROBLEM IV.

To find a second line and point which may be used as the directrix and focus of an ellipse.

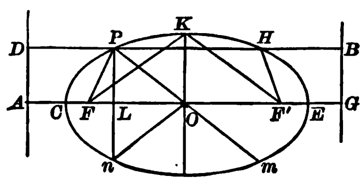


Since the ellipse is symmetrical with respect to its minor axis, if the figure be revolved about KO, AD falling on BG, F on F', and P on H, these points will have the same relative position as before, and hence BG might be used as a directrix and F' as a focus with which to describe the ellipse.

DEFINITION. Any line drawn through the centre of an ellipse, and terminated by the ellipse, is called a *diameter*.

THEOREM XII.

Any diameter of an ellipse is bisected at the centre.



Draw any diameter POm; then $PO = Om$. From P draw the perpendicular PL, and extend it tc meet the ellipse in n;

then $PL = nL$. Revolve the portion at the right about OK, and m will fall at some point of the ellipse on the left of OK, and Om on On. Since

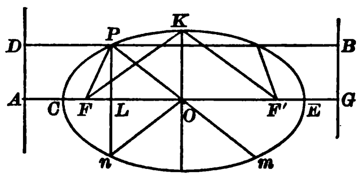
$$mOE = POL = LOn,$$

and since m falls on both the ellipse and the line On, it will fall at n, and

$$Om = On = OP.$$

Theorem XIII.

The sum of the distances from any point on the ellipse to the two foci is constant and equal to the major axis.



Take P any point on the ellipse; we are to prove

$$PF + PF' = CE.$$

Let GB be the second directrix; then (Prob. IV.)

$$DP : PF = BP : PF';$$

which, by composition, is

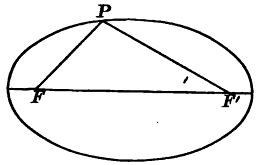
$$DP + BP : PF + PF' = DP : PF.$$

$DP + BP = AG$ at any point, and $DP : PF$ is constant; hence $PF + PF'$ must also be constant; therefore

$$PF + PF' = KF + KF' = 2\,CO,$$

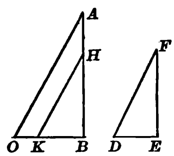
which was to be proved.

SCHOLIUM. The property of the ellipse just proved is the one commonly given as the definition of the curve. It furnishes an easy method of constructing the curve. Thus,



fasten the two ends of a string, longer than FF' at F and F'; move a pencil within the string, so as to keep the string taut; the point will describe an ellipse, since at any point, as P, $PF + PF'$ equals the length of the string.

LEMMA. Two right triangles are similar if the hypothenuse and a side of one are proportional to the hypothenuse and a side of the other.



The two right triangles ABO and DEF, in which

$$AB : AO :: FE : DF,$$

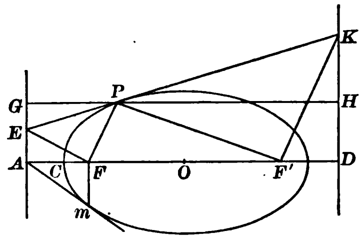
are similar. For, take $BH = EF$, and draw HK parallel to AO; then

$$AB : AO :: HB : HK.$$

Comparing this with the proportion of the hypothesis, we see $HK = DF$, and the triangles HKB and DFE are equal; and as HKB is similar to ABO, DFE is also similar to ABO.

PROBLEM V.

To draw a tangent to an ellipse from a point on the ellipse.



Let P be the given point, connect it to the focus by the line PF, draw EF perpendicular to PF, meeting the directrix in E, join E and P; then EP is the required tangent: for, as EFP is a right angle, the line EP can meet the ellipse in but one point.

COROLLARY I. If Fm is perpendicular to CF', the tangent meets the directrix at A on the major axis extended.

COROLLARY II. Tangents at the extremities of a focal chord meet on the directrix.

SCHOLIUM. The tangent at P might have been drawn by using the focus F' and the directrix KD; hence $PF'K$ is a right angle.

THEOREM XIV.

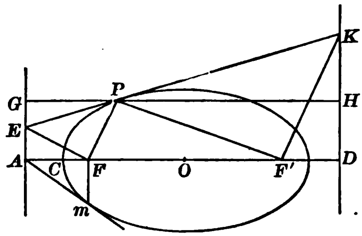
A tangent at any point on an ellipse makes equal angles with lines drawn to the two foci.

Let EP be a tangent to the ellipse; then $EPF = KPF'$. From Problem IV. we learned that

$$GP : PF :: HP : PF',$$

or

$$GP : HP :: PF : PF'.$$



The similar triangles GPE and HPK give

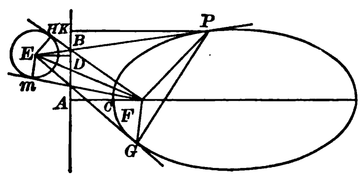
$$GP : HP :: EP : KP;$$

hence $$PF : PF' :: EP : KP.$$

The right triangles EPF and KPF' are therefore similar by the preceding lemma, and the angles EPF and KPF' are equal.

Problem VI.

To draw a tangent to an ellipse from a point without the ellipse.



Let E be the point; draw ED perpendicular to the directrix. Find a line EH, so that

$$AC : CF :: DE : EH,$$

and with EH as a radius, and E as a centre, draw a circle. From F draw FH tangent to this circle, meeting the directrix in B. BE extended will be the tangent required.

For draw PF perpendicular to BF, meeting BE in P; also PK perpendicular to AD; then, by similar triangles,

$$BE : BP :: DE : KP :: EH : PF,$$

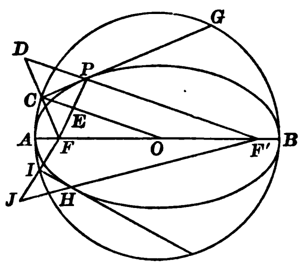
from which $\quad KP : PF :: DE : EH :: AC : CF;$

hence P must be on the ellipse and a point of tangency.

COROLLARY. Draw the tangent EG. Since $EFm = EFH$, adding these to the right angles GFm and HFP, we have the equal angles EFG and EFP; hence a line drawn from the intersection of two tangents to an ellipse to the focus makes equal angles with the focal radii drawn to the point of tangency.

THEOREM XV.

The perpendicular from the focus of an ellipse upon any tangent meets it on the circumscribed circle.



Draw CF perpendicular to the tangent CP, and produce it to meet $F'P$ in D. Since the angle

$$CPF = GPF' = DPC, \quad DP = PF,$$

and $\quad DC = CF;$

hence, as CO bisects the sides of the triangle DFF',

$$CO = \tfrac{1}{2}DF' = \tfrac{1}{2}(PF + PF') = \tfrac{1}{2}AB,$$

and C is a point of the circle on the diameter AB.

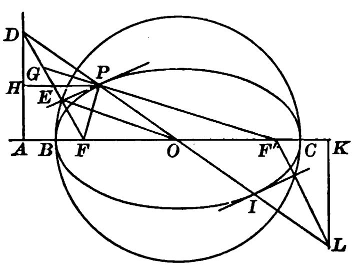
Corollary I. A line from the centre of an ellipse to the point in which a focal perpendicular meets a tangent is parallel to the focal radius from the other focus to the point of tangency.

Corollary II. The same line bisects the focal radius of contact: for $FC = CD$; hence $FE = EP$; and CE, being drawn to the middle of the hypothenuse, equals FE or EP.

Corollary III. Since CP is the perpendicular bisector of DF and IH, of FJ, their intersection is equally distant from D, F, and J. This property of the tangent furnishes a means of drawing a tangent to an ellipse from a point without the ellipse. From the point as a centre, with a radius equal the distance to a focus, draw a circle; from the other focus, with a radius equal the major axis, draw a circle. The intersections of these circles joined to the second focus will pass through the points of tangency required.

Theorem XVI.

If a tangent and a diameter be drawn from a point on an ellipse, a line from the focus perpendicular to the tangent will meet the diameter on the directrix.



Let EP and IP be the tangent and diameter drawn from the point P. Draw EF perpendicular to EP; then EF will

meet IP on the directrix. Represent by D' the point in which EF intersects OP. Since E is on the circle BC (Th. XV.), OE is parallel to $F'PG$, and

$$D'O : D'P :: EO : GP = PF. \qquad (1)$$

Let D be the point in which OP intersects the directrix;

then $$AO : HP :: DO : DP. \qquad (2)$$

But $$AO : BO :: HP : PF; \qquad \text{(Th. VIII.)}$$

or, $$AO : HP :: BO : PF. \qquad (3)$$

From (2) and (3)

$$DO : DP :: BO : PF; \qquad (4)$$

and from (1) and (4)

$$DO : DP :: D'O : D'P;$$

hence D and D' must be the same point.

Corollary. By extending the diameter PI to meet the second directrix in L, it is easily seen that the triangles DFO and $LF'O$ are equal and the lines DF and LF' parallel, and hence that the tangents at P and I, the extremities of a diameter, are parallel.

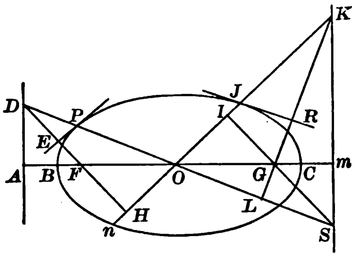
Definition. A diameter drawn parallel to the tangent at the extremity of another diameter is called a *conjugate diameter*.

Theorem XVII.

If one diameter is conjugate to a second, the second is conjugate to the first.

Draw Jn parallel to the tangent EP and produce it to meet the directrix in K; join D, the point in which the diameter PL intersects the directrix with the focus F; then DF is perpendicular to EP and Jn. Join S and the second focus

G. producing the line to meet *Jn* in *I*. *DF* is parallel to *GS* (Cor., Th. XVI.) ; hence *SI* is perpendicular to *Jn*.

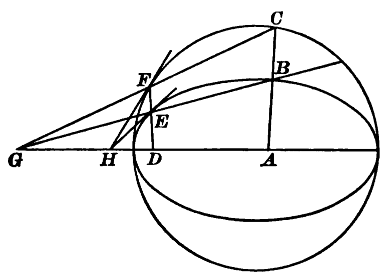


In the triangle *OKS*, *Om* being perpendicular to *KS* and *SI* to *OK*, *KG* must be perpendicular to *OS*. But *KG* is perpendicular to *JR* (Th. XVI.) ; hence the diameter *PL* is parallel to the tangent at the extremity of *Jn*.

COROLLARY. A line from the point in which a diameter intersects the directrix to the focus is perpendicular to the conjugate diameter.

THEOREM XVIII.

A secant of an ellipse and a secant of its circumscribed circle crossing the curves on the same ordinates meet on the major axis, or the major axis extended.



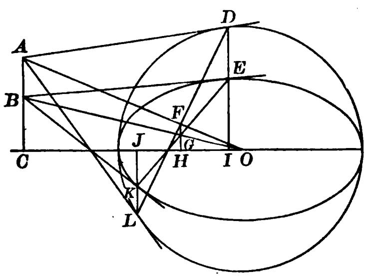
EB and *FC* meet at *G* on the major axis extended. For, major axis : minor axis : : *CA* : *BA* : : *FD* : *ED* ; hence, *FC*, *EB*, and *DA* meet in a point.

COROLLARY. If the secants be revolved about E and F respectively till B comes to E and C to F, we have the tangents HE and HF; hence a tangent to an ellipse and a tangent to its circumscribed circle and tangent on the same ordinate meet on the major axis extended.

This principle affords a convenient method for drawing a tangent to an ellipse at a given point on the ellipse by first drawing an ordinate and a tangent to the circumscribed circle.

THEOREM XIX.

A diameter of an ellipse drawn to the intersection of two tangents bisects the chord joining the points of tangency.



Draw tangents to the ellipse at E and K, and extend the ordinates EI and JK to meet the circumscribed circle in D and L. Draw tangents at D and L, intersecting in A, and from A drop a perpendicular to the major axis, meeting the tangent from E in B, and the tangent from K in B'. Since the tangents from D and E meet on the major axis (Th. XVIII., Cor.),

$$DI : EI :: AC : BC;$$

likewise, the tangents at K and L meeting on the major axis,

$$LJ : KJ :: AC : B'C;$$

but $$DI : EI :: LJ : KJ.$$

From these proportions it appears that the tangents from K and E meet on AC.

Join AO and BO, and from F, the point in which AO intersects DL, drop a perpendicular to the major axis intersecting BO in G; then

$$AC : BC :: FH : GH.$$

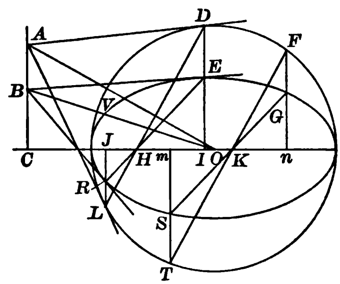
Let G' represent the point in which the perpendicular FH intersects the chord KE. As DL and KE intersect on the major axis (Th. XVIII.),

$$FH : G'H :: DI : EI :: AC : BC;$$

hence G and G' coincide. As DI, FH, and LJ are parallel, and F is the middle of DL, G is the middle of KE.

Theorem XX.

Parallel chords are bisected by the same diameter.



Draw RE and GS, two parallel chords of the ellipse; erect ordinates from their extremities, producing them to meet the circumscribed circle. Join D and L, the extremities of two corresponding ordinates, and through K where GS intersects the major axis draw FT parallel to DL, meeting Gn in F, and mS in T. From the similar triangles thus formed

$$DI : Fn :: DH : FK,$$

$$EI : Gn :: EH : GK,$$

and $DH : FK :: EH : GK$;

hence $DI : Fn :: EI : Gn$,

or $DI : EI :: Fn : Gn$;

and F must be a point of the circumscribed circle (Th. XI.).

Let A be the point in which tangents from D and L intersect, and B the point in which tangents from E and R intersect. As FT is parallel to DL, tangents at F and T will intersect at some point X on AO, and tangents from G and S will intersect on a perpendicular to the major axis from X, and the two intersections will divide this perpendicular in the ratio $DI : EI$ or $AC : BC$ (Th. XIX.); hence the two tangents from G and S intersect on BO, which will bisect GS.

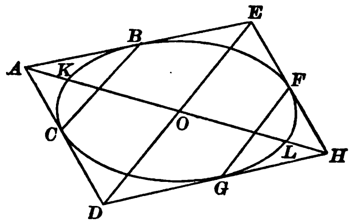
Corollary I. If the parallel chord be moved toward V, the vertex of the diameter BO, the points of intersection with the ellipse approach each other till at V they coincide, and the chord becomes a tangent; hence chords parallel to a tangent at the extremity of a diameter are bisected by that diameter.

Corollary II. A diameter parallel to the tangent is the conjugate to VO; hence chords parallel to a diameter are bisected by its conjugate diameter.

Theorem XXI.

If two pairs of tangents be drawn from two points equally distant from the centre of an ellipse and on the same diameter, they will intersect on the diameter conjugate to that on which the first points are taken.

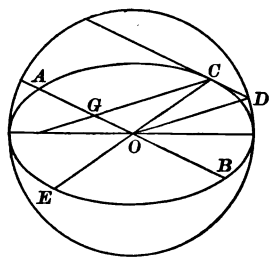
Let the points A and H be taken on the diameter KL equally distant from O; the intersection E of the tangents AB and HF, and D of AC and HG, are on the conjugate diameter to KL.



Revolving the figure half way around on the point O as a pivot, it is easily shown that $DO = EO$. But BC is bisected by AO; hence BC is parallel to DE. BC is also parallel to the tangent at K (Th. XX., Cor. I.), and DE answers the definition of a conjugate diameter to KL.

Theorem XXII.

The distance from the extremity of any diameter to its conjugate measured along a focal radius is constant, and equal to half the major axis.

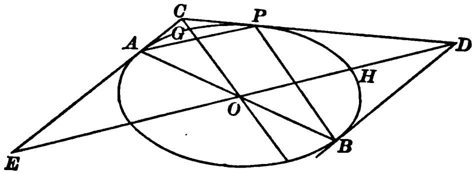


Let CE and AB be two conjugate diameters. Since the focal radius at C is parallel to OD, $GC = OD$.

Definition. Supplemental chords in an ellipse are two chords passing through the opposite extremities of a diameter and intersecting on the ellipse.

THEOREM XXIII.

If a diameter bisects one of two supplemental chords, its conjugate will bisect the other.

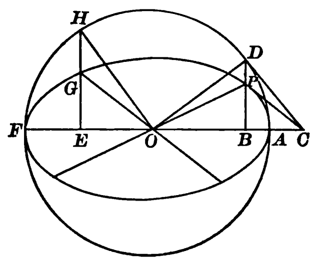


Let AP and PB be two supplemental chords; draw tangents to the ellipse at A, P, and B. From C, the intersection of two of these tangents, draw CO; it will bisect AP (Th. XIX.). From D, the other intersection of the tangents, draw DO; it will bisect PB. The tangents AC and BD at the extremities of the diameter AB being parallel, and $AO = OB$, $EO = OD$. CO bisecting AP and ED, these lines are parallel. But AP is parallel to the tangent at G; hence HO is the conjugate to CO.

COROLLARY. If a chord is parallel to a diameter, its supplemental chord is parallel to the conjugate diameter.

THEOREM XXIV.

The abscissa of the extremity of any diameter of an ellipse is equal to the ordinate of the circumscribed circle passing through the extremity of the conjugate diameter.



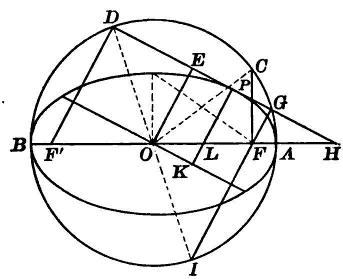
Let OP and OG be conjugate diameters; then OB equals EH. Apply DBC to HEO, so that BC comes on EO, placing C at O; since PC is parallel to GO, P will fall on GO; and, as

$$DB : PB :: HE : GE,$$

D will fall on OH, making the angle DCO equal HOE, and hence DOB equal OHE. The triangle DOB will equal HOE, and $BO = EH$.

Theorem XXV.

If a normal be drawn intersecting the major axis of an ellipse, and the diameter conjugate to that through the point of tangency, the product of its two segments is constant and equals the square of the semi-minor axis.



To prove $PL \cdot PK =$ square of semi-minor axis. It is easily proven that FC equals the semi-minor axis; and, by Geometry,

$$\overline{FC}^2 = FG \cdot FI = FG \cdot F'D.$$

Since PH is the bisector of the exterior angle, and PL of the interior angle of the triangle $F'PF$,

$$HF : HF' :: LF : LF'.$$

By a double composition,

$$HF : HF + LF :: HF + HF' : HF + LF + HF' + LF',$$

or $$HF : HL :: HO : HF'.$$

Because of the parallels $F'D$, OE, PL, and FG,

$$HF : HL :: FG : PL,$$

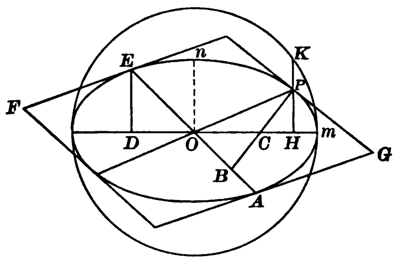
and $$HO : HF' :: OE : F'D;$$

hence $$FG : PL :: OE : F'D,$$

or $$PL \cdot OE = PL \cdot PK = FG \cdot F'D = \overline{FC}^2.$$

THEOREM XXVI.

The area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant, and equals the rectangle on the axes.



Let PC be the normal at P; then the triangles PCH and EOD, having their sides respectively perpendicular, are similar, and

$$PC : OE :: PH : OD :: PH : KH :: On : Om,$$

multiplying the first and last ratios by PB and On, respectively,

$$PC \cdot PB : OE \cdot PB :: \overline{On}^2 : On \cdot Om.$$

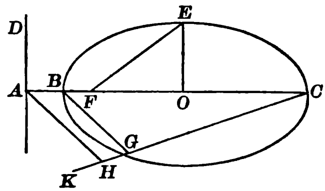
But $$PC \cdot PB = \overline{On}^2; \qquad \text{(Th. XXV.)}$$

hence $$PB \cdot OE = On \cdot Om.$$

The first member represents one-fourth the area of the parallelogram FG, and the second one-fourth the rectangle on the axes.

THEOREM XXVII.

A circle is an ellipse in which the foci are at the centre, the directrix is infinitely distant, and the minor axis equals the major axis.



Draw the ellipse BEC. If B, F, and C are given, A may be found by dividing BC in accordance with the proportion of the definition.

$$BF : BA :: CF : CA.$$

Take on the indefinite line CK, $CH = CF$, and $HG = BF$; draw BG, and HA parallel to BG; then

$$HG : BA :: HC : AC,$$

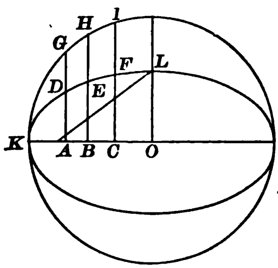
and A is thus fixed. Suppose F to move toward O; the points H and G preserving the relations above assumed will move toward C, and A must thus move from B. When F reaches O, G will be at C, and AH parallel to BC, and the directrix infinitely distant. The foci being at the centre, the distance from any point to either is the same, or half the major axis; consequently, the ellipse has become the circle on the major axis, and the minor axis is evidently equal the major axis.

THEOREM XXVIII.

The area of an ellipse is to the area of the circle formed on its major axis as the minor axis is to the major axis.

Inscribe a regular polygon $KGHI$, etc., in the circle, and from its vertices drop ordinates. Connect the points DEF,

etc., in which these intersect the ellipse. The trapezoids $ABHG$ and $ABED$ having the same altitude,



$$ABED : ABHG :: AD + EB : AG + BH.$$

But $\quad AD : AG :: BE : BH :: LO : KO, \quad$ (Th. XI.)

and $\quad AD + BE : AG + BH :: LO : KO;$

or, $\quad ABED : ABGH :: LO : KO.$

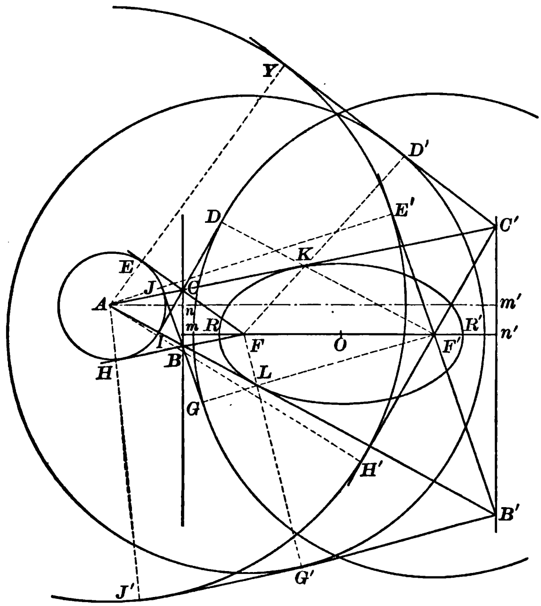
Likewise it may be shown that each trapezoid in the semi-ellipse is to the corresponding trapezoid in the semicircle as $LO : KO$. Taking the sum of the antecedents and the sum of the consequences in the continued proportion thus formed, we find the polygon formed in the ellipse is to the polygon formed in the circle as the minor axis is to the major axis. If the number of sides of the polygon inscribed in the circle be increased indefinitely, the polygon approaches the circle as its limit; likewise the polygon inscribed in the ellipse will approach the ellipse as its limit: hence the area of the ellipse is to the area of the circle formed on its major axis as the minor axis is to the major axis.

Corollary. Writing the theorem as a proportion,

$$\text{area of the ellipse} : \pi\overline{KO}^2 :: LO : KO;$$

hence $\quad \text{area of the ellipse} = \pi KO \cdot LO.$

SCHOLIUM. If from A two tangents AK and AL be drawn and circles for the construction of these tangents be made, AEH with reference to F and BC, and $AE'H'$ with reference to F' and $B'C'$, the difference of their radii is equal the major axis.



For $\qquad AE : An = AE' : Am' = RR' : mn'$,

and $\qquad AE' - AE : Am' - An :: RR' : mn'$,

$$AE' - AE : nm' :: RR' : mn'.$$

As $\qquad nm' = mn'$, $AE' - AE = RR'$.

From F and F' as centres with radii equal to the major axis draw circles $F'DG$ and $FD'G'$. Tangents at D and G, D' and G' pass through C, B, C', and B', the points in which tangents to the ellipse intersect the directrices. Each of these tangents is a tangent to one of the circles used in con-

structing the tangents to the ellipse. Angle $DKC = CKF$; hence

$$KCD = KCF = ECA = ACI.$$

Draw AI perpendicular to DC; the triangles ACI and ACE being equal, $AE = AI$, and I is a point of tangency on the circle AEH. In the same way it might be shown that GBJ, $C'D'Y$, and $B'G'J'$ are common tangents.

Since AE and $D'F$ are both perpendicular to EF, and D' is a point of tangency, AE comes to Y, the point of tangency. Likewise, AJ. AI, and AH come to points of tangency on the circle $AE'H'$.

THE HYPERBOLA.

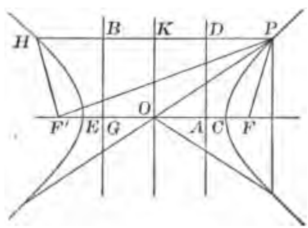
Take F as the focus, and AD the directrix, of a hyperbola; there will be two points on the principal axis, such that

$$AC : CF :: AE : EF :: DP : PF.$$

CE is called the *transverse axis;* O, its *centre*, the *centre*, and any line drawn through O and terminated by the hyperbola, a *diameter*, of the hyperbola.

THEOREM XXIX.

Half the transverse axis of a hyperbola is a mean proportional between the distances from the centre to the focus and to the directrix.



By definition,

$$AC : CF :: AE : EF.$$

Following the same line of reasoning as in Theorem VIII., the student can prove

$$AO : CO :: CO : OF :: AC : CF.$$

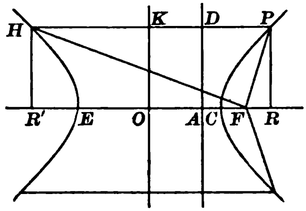
THEOREM XXX.

Ordinates to the hyperbola equally distant from the centre are equal.

Take $OR = OR'$;

then $PR = HR'$.

Let $DP : PF = n$.



As in Theorem IX., the sides PF and RF may be found equal respectively to

$$\frac{RO}{n} - CO \text{ and } RO - \frac{CO}{n},$$

and from these values the perpendicular PR of the right triangle FPR, or

$$PR^2 = \frac{n^2-1}{n^2}(\overline{CO}^2 - \overline{RO}^2).$$

In the right triangle $HR'F$

$$HF = \frac{R'O}{n} + CO,\ R'F = R'O + \frac{CO}{n},$$

and $$\overline{HR'}^2 = \frac{n^2-1}{n^2}(\overline{CO}^2 - \overline{RO}^2);$$

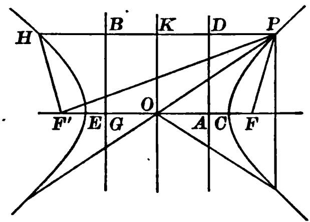
hence PR and HR', being equal to the same thing, are equal to each other.

THEOREM XXXI.

A hyperbola is symmetrical with respect to a perpendicular drawn through the middle of its transverse axis. Proved by revolving half the figure about such a perpendicular.

PROBLEM VII.

To find a second directrix and focus to a hyperbola.



Since the hyperbola is symmetrical with respect to OK, if the figure be revolved on OK, DA will fall on some line GB, F at F', and P on H; these parts having the same relative position as before the revolution, BG and F' might be used as the directrix and focus of the hyperbola.

THEOREM XXXII.

Any diameter of a hyperbola is bisected at the centre. Proof similar to Theorem XII.

THEOREM XXXIII.

The difference of the distances from any point of a hyperbola to the two foci is constant, and equal to the transverse axis.

Let P be any point of a hyperbola whose foci are F and F', and whose directrices are GB and AD (see Fig. above);

then $\qquad PF' - PF = EC.$

For $\qquad BP : PF' :: DP : PF;$

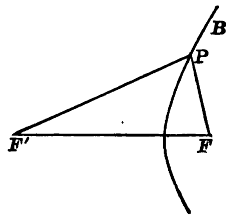
and, by division,

$$BP - DP : PF' - PF :: DP : PF.$$

$DP : PF$ being the same for all points of the hyperbola, and, as $BP - DP = BD$, $PF' - PF$ must be of the same value wherever P is situated; hence

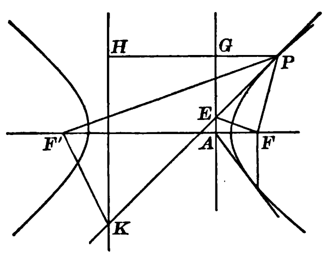
$$PF' - PF = CF' - CF = EC.$$

SCHOLIUM. This property is commonly given as the definition of a hyperbola. It gives a simple method of constructing a hyperbola.



Fasten a ruler so that it may revolve about a point F', to the other end attach a string shorter than the ruler, and fasten its end at F. Revolve the ruler, holding the string tight by a pencil-point P against the ruler; the pencil will describe one branch of a hyperbola. By fastening the ruler at F, and the string at F', the other branch can be made.

PROBLEM VIII.



To draw a tangent to a hyperbola from a point on the hyperbola. Let P be the given point. Join P and F; draw EF perpendicular to PF; then the line EP is the tangent.

Theorem XXXIV.

A tangent at any point of a hyperbola bisects the angle between two lines drawn from the point to the two foci.

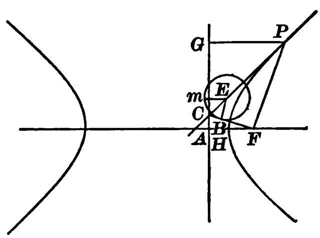
Prove the triangles EPF and KPF' of the last figure similar, as in Theorem XIV.

Problem IX.

To draw a tangent to the hyperbola from a point without the hyperbola. Let E be the point, draw Em perpendicular to the directrix, and with E as a centre and a radius EB such that

$$EB : Em :: HF : AH,$$

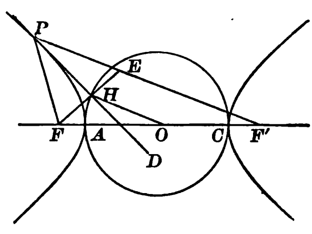
construct a circle.



The entire construction and demonstration are the same as for the corresponding problem in the ellipse.

Theorem XXXV.

The perpendicular from the focus upon any tangent meets it on the circle drawn on the transverse axis as a diameter.



Draw FH perpendicular to the tangent PD;

then $PE = PF$,

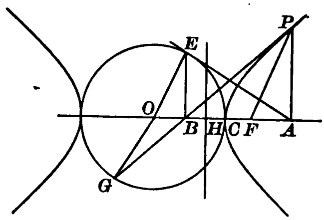
and $EH = HF$;

hence $OH = \frac{1}{2} F'E = \frac{1}{2} CA$.

Corollary. The radius drawn to the point in which a tangent intersects the circle on the transverse axis is parallel to the focal radius of contact.

Theorem XXXVI.

If a tangent to a hyperbola be drawn from a point on the transverse axis, half the transverse axis will be a mean proportional between the distance from the centre to the point, and the abscissa of the point of tangency.



From B draw the tangent BP; then OC is a mean proportional between OB and OA. Since OG is parallel to PF (Th. XXXV., Cor.),

$$OB : BF :: OG : PF :: OC : PF.$$

By Th. XXIX.

$$OH : OC :: HA : PF,$$

or $OH : HA :: OC : PF$;

hence $OB : BF :: OH : HA$,

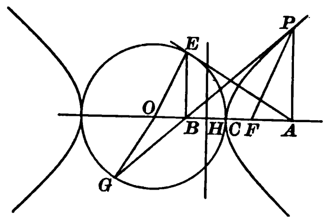
Taking this by composition,

$$OF : OB :: OA : OH;$$

and hence $OB \times OA = OF \times OH = \overline{OC}^2$, (Th. XXIX.) which was to be proved.

Theorem XXXVII.

If from a point on the transverse axis, a tangent be drawn to a hyperbola and an ordinate to the circle on the transverse axis, the tangent to the circle at the extremity of this ordinate will pass through the foot of the ordinate to the point of tangency on the hyperbola.



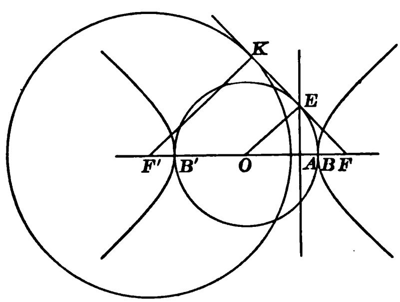
Draw the tangent BP to the hyperbola and the ordinates BE and PA. Join E and A; then, by the previous theorem,

$$OB : OE :: OE : OA,$$

and the triangles OEB and OEA are similar, and OEA is a right angle making EA a tangent to the circle.

Theorem XXXVIII.

If a circle be drawn on the transverse axis of a hyperbola as a diameter, a line connecting its point of intersection with the directrix to the focus is a tangent to the circle.



Let E be the point in which the circle $B'EB$ intersects the

directrix AE; then EF is a tangent to the circle. For OB being a mean proportional between AO and OF,

$$OF : OB :: OB : OA,$$

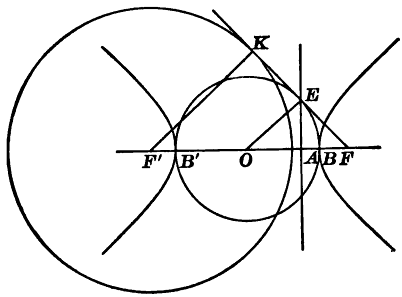
or

$$OF : OE :: OE : OA,$$

and the triangles OFE and OEA must be similar, and, as OAE is a right angle, OEF is also a right angle, or EF is a tangent at E.

Theorem XXXIX.

If a circle be drawn on the transverse axis of a hyperbola as a diameter, and another be drawn with the transverse axis as a radius, and one focus at its centre, the common tangent to these two circles will pass through the other focus.



Draw $F'K$ parallel to OE. Since O is the middle of $F'F$,

$$F'K = 2\,OE = BB',$$

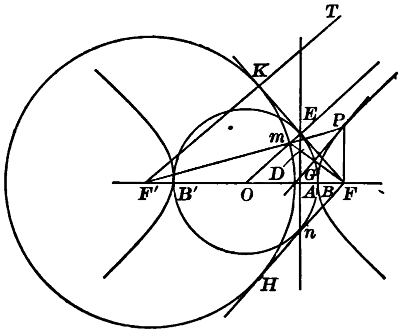
and K would be on a circle having F' as its centre and BB' as its radius; and, as FE is tangent to the circle $B'EB$, it is also tangent to the circle $F'K$.

Theorem XL.

The line joining the middle of the transverse axis to the point in which the circle on the transverse axis intersects the

directrix is the limit to which tangents to the hyperbola approach as the point of tangency is carried from the vertex.

Tangents to the hyperbola approach the position OE as the point of tangency is carried from the vertex B.



Let PG be a tangent. Draw PF and PF'; also the circle $F'KH$ with $F'K = BB'$. Since

$$PF' - PF = BB' = F'm,$$

$Pm = PF$; and PG, bisecting the angle mPF, is the perpendicular bisector of mF. The line mF making with the radius $F'm$ an angle PmF less than a right angle, is within the tangents FK and FH, and will intersect the directrix between E and n, as at D. Suppose the line mF to move toward the tangent KF; the angle PmF approaches a right angle, or the angle TKF and its equal mFP must approach a right angle also; as these angles increase, the point of intersection of their sides recedes until mF becomes KF, when they become parallel, and at this position the tangent, or perpendicular bisector of mF, becomes the perpendicular bisector of KF or the line OE. Hence OE is the limit of tangents as the point of tangency recedes from B.

Corollary. The distance from the centre to the directrix measured along the line OE is equal to the semi-transverse axis.

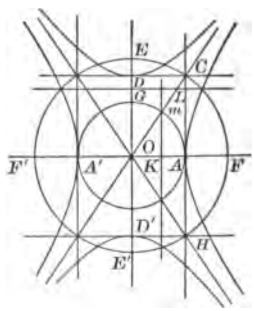
Definition. A limit of tangents to a curve is called an *asymptote*.

CONJUGATE HYPERBOLÆ.

At the vertex A of the transverse axis draw a tangent meeting the asymptotes in C and H. Make $DO = CA$ and $EO = FO$; and, with E as a focus, and OD as the semi-transverse axis, draw a hyperbola. The hyperbola so constructed is called a conjugate hyperbola to the first.

THEOREM XLI.

The asymptotes to a hyperbola are also the asymptotes to the conjugate hyperbola.



Let Km be the directrix of the hyperbola AA', and GL of its conjugate.

As the tangent AC is parallel to Km,

$$OK : Om = OA : OC;$$

but $$OK : OA = OA : OF. \qquad \text{(Th. XXIX.)}$$

m being taken on the asymptote OC, $Om = OA$ (Th. XL., Cor.); hence

$$OC = OF = OE.$$

GL being parallel to DC, and L being taken on OC,

$$OG : OL :: OD : OC.$$

But $$OG : OD :: OD : OE;$$

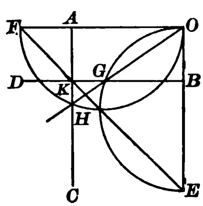
hence $$OL = OD,$$

and the circle with DD', the transverse axis, as its diameter, will intersect the directrix in L, and OL is an asymptote to the hyperbola DD' (Th. XL.).

COROLLARY. The diagonals of the rectangle formed by drawing tangents at the vertices of two conjugate hyperbolæ are asymptotes to the hyperbolæ and equal the distance between the foci of either hyperbola.

THEOREM XLII.

The distance from the centre of a hyperbola to the directrix equals the distance from the focus to the directrix of its conjugate hyperbola.



Let F be the focus, and AC the directrix of a hyperbola, E the focus and BD the directrix of its conjugate. Then OA equals BE.

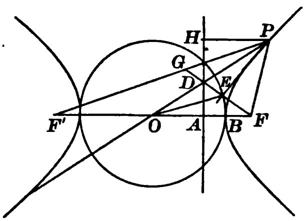
Let G be the point in which DB intersects a circle drawn with OE as a diameter. As OG is a mean proportional between OB and OE, it equals the semi-transverse axis (Th. XXIX.), and hence OG is the asymptote to the hyperbola with E as its focus. Let H be the point in which AC intersects a circle drawn with FO as a diameter; then, as before, OH is the asymptote to the hyperbola with F as its

focus, and OH and OG are the same line (Th. XLI.). The angle HOE intercepting the arcs HO and GE of the equal circles FHO and EGO, these arcs are equal, and $OA = BE$.

COROLLARY. Since FOE and KBE are isosceles right-angled triangles, FE will pass through K; or, the line joining the foci of two conjugate hyperbolæ passes through the intersection of their directrices.

THEOREM XLIII.

A line from the focus of a hyperbola perpendicular to a tangent meets the diameter through the point of tangency on the directrix.



Draw FE from the focus perpendicular to the tangent PE, and let D' represent the point in which it intersects the diameter through OP. As OE is parallel to $F'P$ (Th. XXXV., Cor.), $D'O : D'P :: OE : GP = PF$.

Let D represent the point in which the diameter intersects the directrix;

then $\quad AO : HP :: DO : DP.$

But $\quad AO : BO :: HP : PF,$

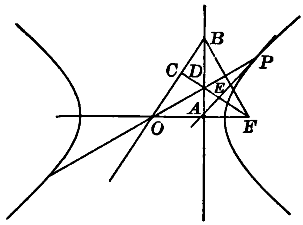
or $\quad AO : HP :: BO : PF :: DO : DP;$

hence $\quad DO : DP :: D'O : D'P;$

and D and D' are the same point.

Theorem XLIV.

A perpendicular from the focus to a diameter of a hyperbola intersects the directrix on a line through the centre and parallel to the tangent at the extremity of the diameter.



Draw OB parallel to the tangent EP, join B and F, let D be the point in which the diameter OP intersects the directrix; then DF is perpendicular to EP or OB, and BA being perpendicular to OF, OP, through their intersection, must be perpendicular to the third side, BF, of the triangle.

As an asymptote is the limit of the tangents to a hyperbola and passes through the centre, a line through the centre and parallel to a tangent will not intersect the hyperbola, but will meet its conjugate.

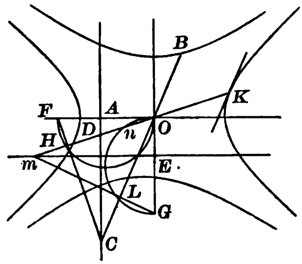
Definition. A diameter of a hyperbola parallel to a tangent to its conjugate hyperbola is called a conjugate diameter to the one passing through the point of tangency.

Theorem XLV.

If one diameter is conjugate to a second, the second is conjugate to the first.

Draw OB parallel to the tangent at K. OB is the diameter conjugate to OK; then OK will be the conjugate diameter to OB. Draw FH perpendicular to OK; it will intersect OB at C on the directrix (Th. XLIV.). Draw GL perpendic-

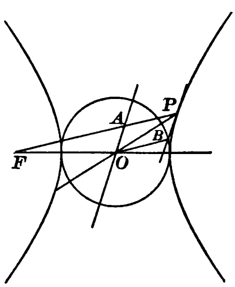
ular to OB, and let it intersect OK in m. On OF and OG as diameters construct semicircles and place the semicircle OG with the lines it contains on the semicircle OF, O on F,



and G on O. As the arcs On and FH measure equal angles, they will coincide, and the line On will fall on FH, m coming somewhere on FC; likewise, GL will come on OL, and hence m must come at C. The perpendicular mE will coincide with CA, and $OE = FA$; hence mE is the directrix of the hyperbola whose focus is G (Th. XLII.). As GL perpendicular to OB intersects OK on the directrix, OK is the conjugate diameter to OB. (Th. XLIV.)

THEOREM XLVI.

The distance from the extremity of any diameter to its conjugate, measured along the focal radius, is constant, and equal to the transverse semi-axis.



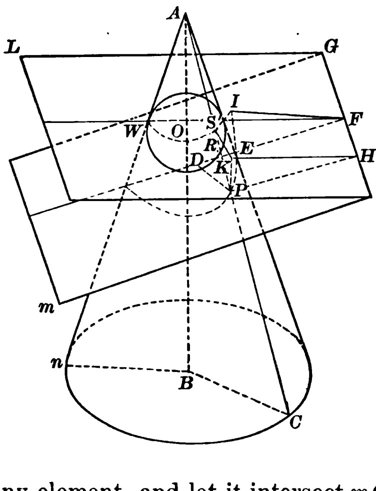
Draw OA parallel to the tangent PB; then PA equals the

transverse semi-axis. Let B be the point in which the tangent PB intersects the circle on the transverse axis. As OB is parallel to FP, OB equals AP.

Theorem XLVII.

Any plane section of a cone of revolution is a conic section.

Let the plane Gm intersect the cone AnC. Inscribe in the cone a sphere OD tangent to the plane Gm at D; and through the circle of contact of the sphere and cone pass the plane LF, intersecting Gm in GH; then will GH be the directrix and D the focus of the conic section in which Gm intersects the cone.



Take AC any element, and let it intersect mG in P; draw PH and DF perpendicular to GH, denoting by E the point in which DF intersects the surface of the cone; also draw PK and EI perpendicular to LF, and intersecting it in K and I respectively. Join IF and KH. From the similar triangles IEF and KPH

$$EF : PH :: EI : KP. \qquad (1)$$

Join KR and SI. Since KP is parallel to AB,

$$KPR = BAC;$$

similarly, $$IES = BAE.$$

But all elements make equal angles with the axis; hence

$$RPK = IES,$$

and the right triangles RKP and SIE are similar, and

$$SE : RP :: EI : KP. \qquad (2)$$

Comparing (1) and (2),

$$SE : RP :: EF : PH.$$

All tangents drawn from a point without a sphere to the sphere are equal; hence $RP = DP$, and $ES = ED$, and

$$ED : DP :: EF : PH,$$

or $$ED : EF :: DP : PH,$$

and P answers the definition of a point on a conic section whose focus is D and directrix GH.

Corollary. If the plane Gm is parallel to an element An,

$$\text{angle } SFE = AWS = ASW = ESF,$$

and $$EF = ES = ED,$$

and the curve is a parabola.

If Gm intersects all the elements below the vertex, then

$$SFE < AWS = ASW = FSE,$$

and $$FE > ES = ED,$$

and the curve is an ellipse.

If Gm intersects some elements above the vertex,

$$SFE > AWS = ASW = FSE,$$

and $$FE < SE = DE,$$

and the curve is a hyperbola.

Exercises.

1. In a parabola, the perpendicular dropped from the focus on any tangent to the curve meets it on the tangent through the vertex.

2. Given the directrix and two points of a parabola, to find the focus.

3. Given the focus and axis of a parabola, to draw the curve so that it shall touch a given straight line.

4. The perpendicular distance from the focus of a parabola to any tangent to the curve is a mean proportional between the distance from the focus to the vertex, and the distance from the focus to the point of contact.

5. To find the locus of the centre of a circle that passes through a given point and touches a given straight line.

6. The ordinate of a point on a parabola is 8, its abscissa 6; find the area included between these and the curve.

7. To draw a tangent to a parabola parallel to a given straight line.

8. Tangents at the extremities of a focal chord intersect at right angles on the directrix.

9. The tangent at any point of a parabola meets the directrix and latus rectum produced in points equally distant from the focus.

10. If the diameter PE meet the directrix in B, and the chord drawn through the focus parallel to the tangent at its vertex in E, prove $PB = PE$.

11. The distance from the focus to the directrix of a parabola is 4. Find the ordinate whose abscissa is 9.

12. The double ordinate through the focus of a parabola is 12; a tangent is drawn to the parabola at a point whose double ordinate is 24. Find how far from the focus the tangent intersects the principal axis.

13. Find the area of the triangle formed by the tangent in the above example, the normal from the same point, and the principal axis.

14. The area of a parabolic segment cut off by a double ordinate is 96, and the corresponding abscissa is 6. Find the distance from the focus to the vertex.

15. Find the locus of the centre of a circle that touches a given straight line and a given circle.

16. The distance from the vertex of the major axis of an ellipse to the directrix is 4, and to the focus is 3. Find the major axis.

17. The major axis of an ellipse is 40, the distance from the focus to the directrix 9. Find the distance from the focus to the centre.

18. The major axis of an ellipse is 10, and the distance between the foci 8. What is the distance from the centre to the directrix? What is the length of the minor axis?

19. In the same ellipse the distance of a point from the focus is 3. What is its distance from the directrix?

20. The ordinate of the circumscribed circle passing through the focus is equal to the minor axis of the ellipse.

21. Lines drawn from the extremities of a diameter of an ellipse to its two foci form a parallelogram.

22. The focal perpendiculars upon any tangent of an ellipse are proportional to the adjacent focal radii of contact.

23. The normal of an ellipse cuts the distance between the foci into segments proportional to the adjacent focal radii.

24. The major axis of an ellipse is 12, the distance between the foci 8. What is the length of a diameter drawn so that one focal radius to its extremity is 5?

25. The minor axis of an ellipse is 6, the sum of two focal radii to a point on the curve is 16. What is the major axis? the distance between the foci? and the area?

26. The two segments of the tangents to an ellipse formed by the circumscribed circle are 15 and 12, the focal perpendicular upon the shorter segment is 5. Find the major axis.

27. In an ellipse whose major axis is 20, a tangent is drawn at a point whose distance from one focus is 12. Find the ratio of the two segments of the tangent formed by the directrices and the point of tangency.

28. If the distance from the centre to the focus in the above example is 8, find the length of the segments of the tangent.

29. If an ordinate be drawn to an ellipse and its circumscribed circle, a parallel to the major axis through the extremity of the ordinate to the ellipse will intersect the radius of the circumscribed circle drawn to the extremity of the ordinate, on a circle drawn with the minor axis as its diameter and the centre of the ellipse as its centre.

30. Lines drawn from the foci bisecting the angles between focal radii drawn to the extremities of a chord meet at the point of intersection of the tangents drawn at the extremities of the chord.

31. Having given the major axis of an ellipse, the foci, and the direction of a diameter, to find its extremities.

32. Having given the focus of an ellipse, its centre, and a tangent, to construct the ellipse.

33. To find the foci of an ellipse, having given the major axis and a tangent to the curve.

34. To find the foci of an ellipse, having given its major axis and a point on the curve.

35. To draw a tangent to an ellipse without using the foci.

36. If a perpendicular be drawn from a focus of an ellipse to any tangent, it is required to find the locus of its intersection with a line drawn from the other focus through the point of tangency.

37. All ellipses described upon a common major axis will have a common subtangent for any given abscissa of contact.

38. The minor axis of an ellipse is 10, and its area 251.328. What is its major axis, and the ordinate through the focus?

39. Lines drawn from the intersection of two tangents to the foci of an ellipse, are respectively perpendicular to the chords joining the intersections of the tangents with the circumscribed circle.

40. Focal perpendiculars upon any tangent meet the directrices on the same diameter.

41. If a tangent be drawn to meet the major axis extended, the semi-major axis will be a mean proportional between the distance from the intersection to the centre, and the abscissa of the point of contact.

42. If a tangent to an ellipse cuts the two directrices, one point of intersection is on the common tangent to two circles drawn from the other intersection and the opposite focus as centres, and with the major axis as the radius of each.

43. The semi-axis minor is a mean proportional between the segment of the major axis formed by either focus.

44. The diagonals of the rectangle formed on the axes of an ellipse are conjugate diameters and equal.

45. In an ellipse whose axes are 8 and 6, what is the length of a diameter that makes an angle of 45° with the major axis? what is the length of its conjugate?

46. In an ellipse whose axes are 60 and 80 a tangent is drawn at a point whose ordinate is 15. Find the length of the tangent between the point of tangency and its intersection with the major axis produced.

47. Diameters drawn through the intersection of any tangent of an ellipse with two parallel tangents are conjugate.

48. A tangent drawn from the point in which a diameter of an ellipse meets the directrix intersects the conjugate diameter on the circumscribed circle.

49. A tangent is drawn intersecting the two tangents at the extremities of the major axis of an ellipse. Show that the circle described on this as a diameter will pass through the foci.

50. The transverse axis of a hyperbola is 3, the distance between the foci is 5. Find the distance from the focus to the directrix, and the length of the ordinate through the focus.

51. Tangents at the extremities of a diameter of a hyperbola are parallel.

52. Diameters which make supplemental angles with the transverse axis of a hyperbola are equal.

53. To draw a tangent to a hyperbola parallel to a given straight line.

54. The perpendicular from either focus to an asymptote is equal to the semi-axis of the conjugate hyperbola.

55. If on the portion of any tangent intercepted between the tangents at the vertices, a circle be described, it will pass through the foci of the hyperbola.

56. The transverse axis of a hyperbola is 8, the distance from the centre to the focus 5. What is the length of the transverse axis of the conjugate hyperbola?

57. If an asymptote meets the directrix in B and the tangent at the vertex in C, then the line from the vertex to B is parallel to the line from the focus to C.

58. The distance between the vertices of two conjugate hyperbolæ equals the distance from the centre to a focus.

59. If an ellipse and a hyperbola have the same foci, tangents drawn to the two curves at their point of intersection are perpendicular to each other.